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Spheromak Power and Helicity Balance

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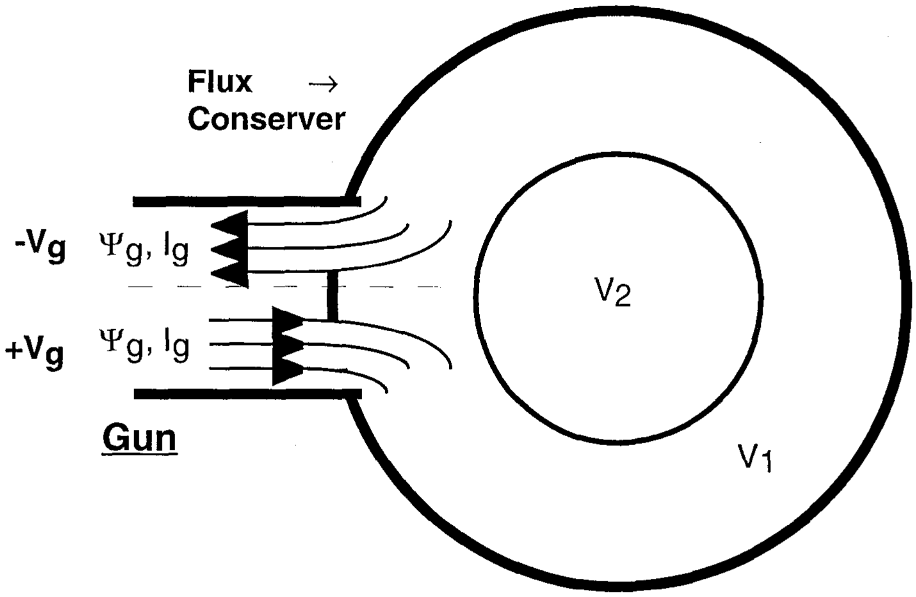
This note addresses the division of gun power and helicity between the open line volume and the closed flux surface volume in a steady state flux core spheromak[1]. Our assumptions are that fine scale turbulence maintains each region close to a Taylor state, $\mu_o \mathbf{j} = \lambda \mathbf{B}$.

The gun region that feeds these two volumes surrounded by a flux conserver is shown topologicaly below. (The actual geometry is toroidal). Flux and current from the magnetized gun flow on open lines around the entire closed surface containing the spheromak. The gun current flows down the potential gradient, the potential difference between the two ends of each line being the gun voltage. Here, the gun voltage excludes the sheath drops at each end.

When these volumes have different values of λ (ratio of $\mu_o B^{-2} \mathbf{j} \bullet \mathbf{B}$ in each region) in the open line volume V_1 and the closed spheromak volume V_2 the efficiency of transferring the gun power to the spheromak to sustain the ohmic loss is the λ-ratio of these regions, in the limit $V_1 << V_2$. This result follows immediately from helicity balance in that limit. Here we give an accounting of all the gun power, and do not assume a small edge (open line) region.



Helicity Conservation

The rate of loss of helicity (K) in the spheromak (volume 2) and ohmic power loss there are related. Using subscript s for the spheromak,

$$\frac{dK_s}{dt} = \int 2E \bullet B dV \equiv \frac{K_s}{\tau_h} = \frac{2\eta\lambda_s^2}{\mu_o} K_s$$

$$P_{oh} = \int E \bullet j dV = \frac{\lambda_s}{2\mu_o} \int 2E \bullet B dV = \frac{\lambda_s}{2\mu_o} \frac{dK_s}{dt}$$

We've assumed λ is constant to take it outside the power integral of E•j, and ignored + and - signs because it is understood when helicity and power are gained or lost. Using subscript g for the gun, it provides helicity at the rate,

$$\frac{dK_g}{dt} = 2V_g\Psi_g = \frac{2\mu_o}{\lambda_g} P_{gun} \equiv \frac{K_g}{\tau_o}$$

defining the time τ_o. If volume 1 is small, by conservation of helicity $\left(\frac{dK_g}{dt} = \frac{dK_s}{dt}\right)$ we find

$$P_{oh} = \frac{\lambda_s}{\lambda_g} P_{gun}$$

If volume 1 is not small we treat each region using distinct values of resistivity η, helicity K, and λ. We assume $\mu_o J_1 = \lambda_1 B_1$ and $\mu_o J_2 = \lambda_2 B_2$ and that the gun feeds region 1. Our model is oversimplified by assuming constant λ-values in each region and a step function drop from region 1 to region 2. The helicity loss and ohmic powers are;

$$\frac{dK_1}{dt} = \frac{K_1}{\tau_1} = \frac{2\mu_o}{\lambda_1}(P_{oh})_1 \qquad \text{and} \qquad \frac{dK_2}{dt} = \frac{K_2}{\tau_2} = \frac{2\mu_o}{\lambda_2}(P_{oh})_2$$

with $\tau_\alpha = \frac{\mu_o}{2\eta_\alpha\lambda_\alpha^2}$. The gun supplies the total helicity to maintain a steady state and its generation time constant is τ_o;

$$\frac{dK_{tot}}{dt} = 2V_g\Psi_g = \frac{2\mu_o}{\lambda_1} P_{gun} \equiv \frac{K_1 + K_2}{\tau_o}$$

Since helicity is conserved,

$$\frac{K_1 + K_2}{\tau_o} = \frac{K_1}{\tau_1} + \frac{K_2}{\tau_2}$$

This equation determines τ_o,

$$\frac{1}{\tau_o} = \frac{1}{K_1 + K_2}\left\{\frac{K_1}{\tau_1} + \frac{K_2}{\tau_2}\right\}$$

Power Balance

We can use this result to compute the ratios of ohmic power and gun power;

$$F_1 \equiv \frac{(P_{oh})_1}{P_{gun}} = \frac{\tau_o}{\tau_1}\left\{\frac{K_1}{K_1 + K_2}\right\} = \left[1 + \frac{K_2\tau_1}{K_1\tau_2}\right]^{-1}$$

$$F_2 \equiv \frac{(P_{oh})_2}{P_{gun}} = \frac{\tau_o}{\tau_2}\frac{\lambda_2}{\lambda_1}\left\{\frac{K_2}{K_1 + K_2}\right\} = \frac{\lambda_2}{\lambda_1}\left[1 + \frac{K_1\tau_2}{K_2\tau_1}\right]^{-1}$$

The sum is

$$F_1 + F_2 = \frac{1}{K_1\tau_2 + K_2\tau_1}\{K_1\tau_2 + \frac{\lambda_2}{\lambda_1}K_2\tau_1\}$$

If the volume of region 1 shrinks to zero ($K_1 = 0$, $\tau_o = \tau_2$) the ohmic power in region 2 is $\frac{\lambda_2}{\lambda_1}P_g$, our first result. Here, $K_\alpha = \int \frac{B_\alpha{}^2}{\lambda_a} dV_a$.

Now let us balance the gun power with losses inside the closed volume. In what follows we assume there is a non-zero mean value of the products $\mathbf{E}\bullet\mathbf{B}$ and $\mathbf{j}\bullet\mathbf{E}$ which determine helicity loss and power loss respectively. First, the flow of power into region 1 is along the open lines, where there is an electric field consisting of the dynamo field $\mathbf{E}_{dyn1}$ (which may or may not time average to zero) and the ohmic field $\eta_1\mathbf{j}_1$. There is a flow of power across the separatrix surface P_2 that feeds the dynamo in that region. So we can equate the inward flow of power from the gun to loss of power in the volume plus flow of power out of region 1 into region 2;

$$P_g = \int \mathbf{j}_1\bullet\mathbf{E}_1 dV_1 + P_2 = \int \mathbf{j}_1\bullet\mathbf{E}_{\mathbf{dyn1}} dV_1 + \int \eta_1 j_1{}^2 dV_1 + P_2$$

That dynamo in region 2 sustains the field against ohmic losses in region 2 that would otherwise cause the stored energy to decay.

$$P_2 = \int \mathbf{j}_2\bullet\mathbf{E}_{\mathbf{dyn2}} dV_2 = \int \eta_2 j_2{}^2 dV_2$$

Adding these and using the fractions of gun power going to ohmic heating,

$$P_g(1 - F_1 - F_2) = \int \mathbf{j}_1\bullet\mathbf{E}_{\mathbf{dyn1}} dV_1$$

By this construct, the balance of the gun power, above the resistive loss by electron flow, goes to the dynamo in region 1. This power first goes into waves or MHD modes but eventually into the plasma ions and/or electrons according to details of the processes that try to maintain a Taylor state.

The powers above can be viewed as inputs; the gun power provides the input for dynamo power and ohmic heat. One can also write diffferent equations that distribute this input heat and wave power to various loss channels, such as radiation, power to restore charge exchange ion losses, conduction or convection loss, etc. One needs to understand the dynamo power in more detail to be able to write a detailed power balance in region 1. Which losses are driven by the collisions of electrons whose flow is sustained by the potential difference between the gun and dynamo, and which are driven by the fully evolved dynamo? In region 2, the dynamo power to the plasma is assumed to be converted fully to maintaining the ohmic current, which subsequently supplies the power for the various loss channels within that volume.

Dynamo Transport

We assumed so far that the λ-values were constant in each region, with an infinite gradient at the boundary of the two regions. In reality there is some gradient everywhere, and integrals involving $\mathbf{j}$ use $\mu_o^{-1}\lambda\mathbf{B}$ so that λ cannot be taken outside of those integrals as we have done. To understand better the role of the λ-gradients let us use the mean of a product of the dynamo electric field and magnetic field suggested by Hooper[2] from the work of Boozer[3] and Strauss[4]. This suggestion is valid for small-amplitude fluctuations, and is a generic model of the dynamo containing the λ-gradient and κ, a hyper-resistivity; $\mathbf{E}_{dyn}\bullet\mathbf{B} = -\nabla \bullet \{\frac{\kappa B^2}{\mu_o}\nabla\lambda\}$. Then, a calculation of the dynamo power loss in a volume V is

$$\int \mathbf{j}\bullet\mathbf{E}_{dyn}dV = \int \frac{\lambda}{\mu_o}\mathbf{B}\bullet\mathbf{E}_{dyn}dV = -\int \frac{\lambda}{\mu_o}\nabla \bullet \{\frac{\kappa B^2}{\mu_o}\nabla\lambda\}dV$$

$$= -\int \nabla \bullet \frac{\lambda}{\mu_o}\{\frac{\kappa B^2}{\mu_o}\nabla\lambda\}dV + \int \{\frac{\kappa B^2}{\mu_o}\nabla\lambda\}\bullet \frac{\nabla\lambda}{\mu_o}dV$$

If we first apply this result to the entire volume inside the flux conserver, the first integral can be converted to a surface term which is zero on the walls since $\nabla\lambda$ is zero there. There are two wall surfaces, one where the gun flux enters the volume and the remainder where the flux is parallel to the wall. In either region, $\nabla\lambda$ is zero at the wall. The remaining term can be written $\int \kappa j^2 \frac{[\nabla\lambda]^2}{\lambda^2}dV$, which suggests that the strength of the dynamo in a given spot is inversely proportional to the square of the gradient length there.

Next we apply this to volume 1, where there are two kinds of surfaces, the flux conserver and the separatrix surface between regions. The surface integral is not zero on the latter, so that our power P_2 that flows into region 2 is proportional to $\nabla\lambda$ on that surface. With S_1 the common surface connecting the two regions,

$$P_2 = \oint \frac{\lambda_1}{\mu_o}\{\frac{\kappa B^2}{\mu_o}\nabla\lambda_1\}dS_1 \quad \text{and} \quad P_{dyn1} = \int \kappa j_1^2 \frac{[\nabla\lambda_1]^2}{\lambda_1^2}dV_1$$

If there is a gradient on the surface S_1 then there will a dynamo power P_{dyn2} that was negelected above in equating P_2 to $(P_{oh})_2$.

These results are presented only to qualitatively understand power density and flow as they relate to λ-gradients. The concept of hyper-resistivity may not be fully applicable here. And, we point out the obvious, that the calculations of this note are for axisymmetric ideal spheromaks kept at the Taylor state by fine grain turbulence. The calculations are done for the purpose of better understanding the gun power balance, not for understanding the physics of spheromaks driven by large amplitude low mode number (both axisymmetric and non-axisymmetric) instabilities.

References

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Appendix

We've used the relationship between power and helicity rate for regions of constant λ , $P_{oh} = \frac{\lambda}{2\mu_o}\frac{dK}{dt}$ dependent on the guage invariant form of helicity,

$$K_o = \int \mathbf{A} \bullet \mathbf{B} dV = \int \lambda^{-1} \mathbf{B} \bullet \mathbf{B} dV = \lambda^{-1}(2\mu_o W_{mag}) \qquad (\lambda = \text{constant})$$

In steady state the loss of helicity and the rate at which magnetic energy is converted to ohmic losses can be equated, $\frac{\lambda}{2\mu_o}\frac{dK_o}{dt} = \frac{dW_{mag}}{dt} = P_{oh}$.

Although the separatrix boundary between regions 1 and 2 is not a conductor, we assume that the mean magnetic fields lie in flux surfaces so that $\mathbf{B} \bullet \mathbf{n} = 0$ on that surface. The regions are simply connected and the helicity K_o is gauge invariant, and helicity can be defined in each region.

Nonetheless, we could use another form for helicity which is always gauge invariant[1], $K = K_o - \oint A \bullet d\ell_p \oint A \bullet d\ell_T$. Here, the first closed path integral is the short way (poloidal) around a flux surface boundary ($B_n = 0$), and the second is the long way around (toroidal). Using the separatrix boundary surface to evaluate them, the toroidal flux inside the separatrix (region 2) is $\Phi_T = \lambda \oint A \bullet d\ell_p$. Also, $\lambda \oint A \bullet d\ell_T = \oint B \bullet d\ell_T = 2\pi R B_T$ (note that RB_T is constant, so the toroidal path can be taken anywhere on the flux surface). Now, though λK and $2\mu_o W_m$ are related differently than is λK_o, our integrals are constant during equilibrium so $\frac{dK}{dt}$ can be related directly to ohmic power.

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